

HIGHER ORDER HOCHSCHILD (CO)HOMOLOGY OF NONCOMMUTATIVE ALGEBRAS

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ABSTRACT. Hochschild (co)homology of Hochschild and higher order Hochschild (co)homology of Pirashvili are useful tools for a variety of applications including deformations of algebras. At the introduction of higher order Hochschild (co)homology, we can consider the (co)homology of any commutative algebra with symmetric coefficient modules, however traditional Hochschild (co)homology is able to be computed for any associative algebra with not necessarily symmetric coefficient modules. In a previous paper, the author generalized higher order Hochschild cohomology for multimodule coefficients (which need not be symmetric). In the current paper, we continue to generalize higher order Hochschild (co)homology to work with associative algebras which need not be commutative.

INTRODUCTION

Given a field, k , k -algebra, A and an A -bimodule, M , we can associate a chain complex $C_\bullet(A, M)$, where

$$C_n(A, M) := M \otimes A^{\otimes n}$$

and differentials are given by

$$\delta_n = \sum_{i=0}^n (-1)^i d_i$$

where $d_i: M \otimes A^{\otimes n} \rightarrow M \otimes A^{\otimes n-1}$ are defined as follows:

$$d_i(m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \begin{cases} ma_1 \otimes a_2 \otimes \cdots \otimes a_n & i = 0 \\ m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 1 \leq i \leq n-1 \\ a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n \end{cases}$$

The homology of the above chain complex was introduced by Hochschild in [9] and is referred to as the Hochschild homology of A with coefficients in M . It can be observed that the construction above has a simplicial description. For the simplicial description, let A be a commutative k -algebra and M be a symmetric A -bimodule and consider the Loday functor $L(A, M)$ from the category of finite pointed sets, Γ to the category of k -modules, $k\text{-mod}$ (see [11] and [13]), given by

$$L(A, M): \Gamma \rightarrow k\text{-mod}$$

$$V \rightarrow M \otimes A^{\otimes |V|-1}$$

2010 *Mathematics Subject Classification.* Primary 16E40, Secondary 16S80, 18G30, 55U10.
Key words and phrases. Hochschild, cohomology, homology, higher order, multimodule.

for objects $V \in \Gamma$. For morphisms $\varphi: V_1 \rightarrow V_2 \in \Gamma$ we identify $V_1 = m_+ = \{0, 1, \dots, m\}$ and $V_2 = n_+ = \{0, 1, \dots, n\}$ (where $|V_1| = m + 1, |V_2| = n + 1$ and 0 is the fixed element) and let

$$L(A, M)\varphi(m \otimes a_1 \otimes \cdots \otimes a_m) = (b_0 m \otimes \cdots \otimes b_n)$$

where

$$b_i = \prod_{\{j \in m_+ | j \neq 0, \varphi(j)=i\}} a_j.$$

Now if we consider the minimal decomposition of the pointed simplicial set $S_\bullet^1: \Delta^{op} \rightarrow \Gamma$ (with one non-degenerate 1-simplex) then we have a simplicial k -module

$$\Delta^{op} \xrightarrow{S_\bullet^1} \Gamma \xrightarrow{L(A, M)} k - mod$$

which gives the Hochschild chain complex when we define differentials to be alternating face maps.

Similarly we have a functor [8]

$$\mathcal{H}(A, M): \Gamma \rightarrow k - mod$$

$$n_+ \rightarrow \text{hom}(A^{\otimes n}, M)$$

where for a map $\varphi: m_+ \rightarrow n_+$ and map $f: A^{\otimes n} \rightarrow M$ we have

$$\mathcal{H}(A, M)\varphi(f)(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = b_0 f(b_1 \otimes \cdots \otimes b_n)$$

where

$$b_i = \prod_{\{j \in m_+ | j \neq 0, \varphi(j)=i\}} a_j.$$

The associated cochain complex $\Delta^{op} \xrightarrow{S_\bullet^1} \Gamma \xrightarrow{\mathcal{H}(A, M)} k - mod$ is the Hochschild cochain complex, whose resulting homology is known as Hochschild cohomology of A with coefficients in M .

It was realized by Pirashvili in [13] that for any simplicial set X_\bullet , we can consider the chain complex

$$\Delta^{op} \xrightarrow{X_\bullet} \Gamma \xrightarrow{L(A, M)} k - mod$$

or similarly the cochain complex (see [8])

$$\Delta^{op} \xrightarrow{X_\bullet} \Gamma \xrightarrow{\mathcal{H}(A, M)} k - mod.$$

The resulting homologies are referred to as higher order Hochschild homology and higher order Hochschild cohomology respectively.

Hochschild (co)homology and higher order Hochschild (co)homology have been shown to be incredibly useful tools for a variety of concepts. Hochschild cohomology and higher order Hochschild cohomology have been used to study deformations of algebras and modules (for example see [5], [6], [7], [10], [11], [14] and [15]). In 1976, Dennis developed a map from K-theory to Hochschild homology which was benefited by the introduction of topological Hochschild homology in [1] by Bökstedt in 1985 as an answer to Goodwillie's conjecture (see [4]).

One thing that can be noticed when considering the history of Hochschild cohomology is that traditional Hochschild (co)homology is of any associative k -algebra, A and with coefficients in any A -bimodule, M , however higher order Hochschild (co)homology was restricted to commutative k -algebras with coefficients in symmetric A -bimodules. In [2] the author generalizes the higher order Hochschild

construction to have coefficients in not necessarily symmetric multimodules. The modules able to be used depend heavily on the simplicial sets that the (co)chain complexes are built over. In this paper we aim to generalize higher order Hochschild (co)homology to work with not necessarily commutative algebras and not necessarily symmetric multimodules. In particular, we will demonstrate the “maximal” algebraic structure allowed by simplicial sets when constructing higher order Hochschild (co)homology. Our construction for all simplicial sets is in line with Pirashvili and Richter’s construction in [12] where they gave a description of a simplicial noncommutative circle as a functor from the simplicial category, Δ^{op} to the category of noncommutative sets, which allowed them to construct Hochschild (co)homology (as well as other homologies of functors) with not necessarily commutative algebras and not necessarily symmetric bimodules.

1. PRELIMINARIES

In this paper we fix a field k and denote \otimes_k by \otimes . We assume that the reader has some familiarity with simplicial sets and Hochschild (co)homology, but we provide a quick introduction of simplicial sets here and hope that the introduction along with an extra explanation below provide a sufficient description of Hochschild (co)homology.

1.1. Simplicial Sets. Let Δ^{op} be the category whose objects are indexed by the natural numbers and morphisms are generated by face maps

$$\{d_i: [n] \rightarrow [n-1] | 0 \leq i \leq n\}$$

and degeneracy maps

$$\{s_i: [n] \rightarrow [n+1] | 0 \leq i \leq n\}$$

with the following simplicial relations:

$$\begin{cases} d_i d_j = d_{j-1} d_i & i < j \\ d_i s_j = s_{j-1} d_i & i < j \\ d_i s_j = id & i = j \text{ or } i = j+1 \\ d_i s_j = s_j d_{i-1} & i > j+1 \\ s_i s_j = s_{j+1} s_i & i \leq j \end{cases}$$

Let \mathbf{Set} be the category of small sets. We have the following definitions.

Definition 1.1. A *simplicial set* is a functor $X_\bullet: \Delta^{op} \rightarrow \mathbf{Set}$

Definition 1.2. A *simplicial object* in a category C is a functor $X_\bullet: \Delta^{op} \rightarrow C$

Definition 1.3. A *cosimplicial object* in a category C is a contravariant functor $X_\bullet: \Delta^{op} \rightarrow C$

1.2. Higher Order Hochschild Cohomology. In [2] the author uses a generalized version of higher order Hochschild cohomology which allows multimodule coefficients. In order to allow more actions for the coefficient modules, we need to make action identifications in to get a cosimplicial k -module.

Theorem 1.4. [2, 1.1] *Let A be a commutative k -algebra. Given a pointed simplicial set X_\bullet , there exists a cosimplicial k -module $(M, X)^\bullet$ associated to an A -module M given by*

$$(M, X)^n = \text{hom}_k(k \otimes_k \bigotimes_{\substack{\sigma \in X_n \\ \sigma \neq *}} A, M)$$

with coface and codegeneracy maps given by

$$d_n^i f(1 \otimes_k \bigotimes_{\substack{\sigma \in X_{n+1} \\ \sigma \neq *}} a_\sigma) = \prod_{\substack{\sigma \in X_{n+1} \\ d_i(\sigma) = *}} (\Lambda_{(i,n)}^\sigma(a_\sigma)) \cdot f(1 \otimes_k \bigotimes_{\substack{\Omega \in X_n \\ \Omega \neq *}} \prod_{\substack{\sigma \in X_{n+1} \\ d_i(\sigma) = \Omega}} a_\sigma)$$

and

$$s_n^i f(1 \otimes_k \bigotimes_{\substack{\sigma \in X_{n+1} \\ \sigma \neq *}} a_\sigma) = f(1 \otimes_k \bigotimes_{\substack{\Omega \in X_{n+1} \\ \Omega \neq *}} 1 \cdot \prod_{\substack{\sigma \in X_n \\ s_i(\sigma) = \Omega}} a_\sigma)$$

if the actions $\Lambda_{(-,-)}^-$ on M satisfy the following for simplices σ, Ω and μ :

- i) $\Lambda_{(j,n+1)}^\sigma = \Lambda_{(i,n+1)}^\sigma$ if $\sigma \neq *, d_i(\sigma) = d_j(\sigma) = *$ and the dimension of σ is at least 2 and $i < j$.
- ii) $\Lambda_{(j,n+1)}^\sigma = \Lambda_{(j-1,n)}^\Omega$ if $d_i(\sigma) = \Omega, d_j(\sigma) = *, d_{j-1}(\Omega) = *$ and the dimension of σ is at least 2.
- iii) $\Lambda_{(i,n)}^\Omega = \Lambda_{(j-1,n)}^\mu$ if $d_i(\Omega) = *, d_{j-1}(\mu) = *$ and there exists a σ of dimension at least 2 where $d_j(\sigma) = \Omega, d_i(\sigma) = \mu$ and $i < j$.
- iv) $\Lambda_{(i,n)}^\Omega = \Lambda_{(i,n+1)}^\sigma$ if $d_i(\sigma) = *, d_i(\Omega) = *, d_j(\sigma) = \Omega$ and the dimension of σ is at least 2.

where $\Lambda_{(i,n)}^\sigma(a)$ represents the $(\Lambda_{(i,n)}^\sigma)$ action of $a \in A$ whenever $0 \leq i \leq 1, \sigma \in X_{n+1}$ and $d_i(\sigma) = *$. We take a product of such actions to represent the composition of the actions, which we assume to be commutative (i.e. if M has two actions, we actually assume that M is a bimodule).

1.3. Organization of Paper. In Section 2 we describe how to construct the higher order Hochschild cohomology cochain complex for algebras which are not necessarily commutative. We do so by describing the necessary characteristics a simplicial set must have in order to work with noncommutative algebras, which amounts to considering “simplicial orderings” In Section 3 we note that the construction in this paper can be extended to higher order Hochschild cohomology of pairs of simplicial sets from [3].

2. NOT NECESSARILY COMMUTATIVE ALGEBRAS

We would like to develop higher order Hochschild (co)homology to accept non-commutative algebras, but in order to do so we need to provide an order in which to multiply elements in the equation from Theorem 1.4. In the following section we will determine algebras and modules allowed for higher order Hochschild cohomology and leave it to the reader to check that the same holds for homology.

Definition 2.1. Given a simplicial set X_\bullet and a map $f: X_n \rightarrow X_m$, for a maximal subset $S = \{\sigma_i\}_{i \in I}$ of X_n with the property that $f(\sigma_i) = f(\sigma_j)$ for all $i, j \in I$ (by maximal we mean that if $\tau \in X_n$ so that $f(\tau) = f(\sigma_i)$ for some $i \in I$ then $\tau \in S$) then we refer to an ordering on S as an *f-simplicial ordering* of S .

Now if we consider all d_i simplicial orderings on the corresponding subsets of X_n we see that this gives a way to multiply elements of our algebra for each d_i if A is not necessarily commutative (where products of elements in each tensor factor are multiplied with elements represented by smaller simplices on the left and elements represented by larger simplices on the right). The main issue is that in order for the associated cosimplicial k -module to satisfy the cosimplicial identities, we need the d_i simplicial orderings to have some compatibilities.

Definition 2.2. Given a simplicial set X_\bullet if the $d_{i_k} d_{i_{k-1}} \cdots d_{i_1} : X_n \rightarrow X_{n-k}$ simplicial ordering agrees with the $d_{i_k} d_{i_{k-1}} \cdots d_{i_{k-j+1}} : X_{n-j} \rightarrow X_{n-k}$ simplicial ordering for all possible compositions of face maps and all corresponding subsets of X_n and X_{n-j} then we say X_\bullet admits a *not necessarily commutative multiplicative ordering* (which we denote NNCMO). By agree, we mean that if $\sigma_1 < \sigma_2 < \dots < \sigma_r$ in the $d_{i_k} d_{i_{k-1}} \cdots d_{i_1}$ simplicial ordering, then since

$$\begin{aligned} d_{i_k} d_{i_{k-1}} \cdots d_{i_{k-j+1}} (d_{i_{k-j}} \cdots d_{i_1} (\sigma_1)) &= d_{i_k} d_{i_{k-1}} \cdots d_{i_{k-j+1}} (d_{i_{k-j}} \cdots d_{i_1} (\sigma_2)) = \\ &= \dots = d_{i_k} d_{i_{k-1}} \cdots d_{i_{k-j+1}} (d_{i_{k-j}} \cdots d_{i_1} (\sigma_r)), \end{aligned}$$

we expect the same ordering from the $d_{i_k} \cdots d_{i_{k-j+1}}$ simplicial ordering i.e.

$$d_{i_k} \cdots d_{i_{k-j+1}} (\sigma_1) < d_{i_k} \cdots d_{i_{k-j+1}} (\sigma_2) < \dots < d_{i_k} \cdots d_{i_{k-j+1}} (\sigma_r).$$

We now have the following Theorem.

Theorem 2.3. Given a simplicial set X_\bullet , non-commutative algebra A and A -multimodule (whose left actions can also be considered right actions) M , the Hochschild cosimplicial k -module $(A, M, X)^\bullet$ exists if and only if X_\bullet admits an NNCMO.

Proof. From [2] we see that if A is a commutative algebra, then $(A, M, X)^\bullet$ exists. The only thing that needs to be checked if A is non-commutative is that the cosimplicial identities can be satisfied when we order the elements to be multiplied via the NNCMO. We provide a proof by contradiction. Suppose there is an issue. i.e. for some $d_{i_k} \cdots d_{i_1} = d_{j_k} \cdots d_{j_1} \in \Delta^{op}$ and $\sigma_1, \sigma_2 \in X_n$ we get that $\delta^{i_1} \cdots \delta^{i_k} f(\otimes_{\sigma \in X_n} a_\sigma)$ and $\delta^{j_1} \cdots \delta^{j_k} f(\otimes_{\sigma \in X_n} a_\sigma)$ have a_{σ_1} and a_{σ_2} in the same tensor factor, but in different orders. Let the common tensor factor be represented by $\tau \in X_{n-k}$, then $d_{i_k} \cdots d_{i_1} (\sigma_1) = d_{j_k} \cdots d_{j_1} (\sigma_1) = \tau = d_{i_k} \cdots d_{i_1} (\sigma_2) = d_{j_k} \cdots d_{j_1} (\sigma_2)$. This gives two different simplicial orderings of σ_1 and σ_2 which must agree which provides us with a contradiction. \square

For a commutative algebra A and A -module M , it can be seen that any left action is also by definition a right action (we will denote such an action as an *lr action*), but for non-commutative algebras, coefficient modules need not have lr actions. To see what actions on coefficient modules need not be lr actions we consider the following proposition.

Proposition 2.1. For the cosimplicial k -module $(A, M, X)^\bullet$ suppose there exists $\sigma < \tau \in X_n$ with $d_{i_k} \cdots d_{i_1} (\sigma) = d_{i_k} \cdots d_{i_1} (\tau) = \omega \neq *$ with $d_{i_r} \cdots d_{k+1} (\omega) = *$ and $d_{r-1} \cdots d_{k+1} (\omega) \neq *$ then $\Lambda_{(i_r, n-r)}^{d_{r-1} \cdots d_{k+1} (\omega)}$ is a

- left action if there exists a set of maps $d_{j_r} \cdots d_{j_1} = d_{i_r} \cdots d_{i_1}$ with the property that $d_{j_{r-l}} \cdots d_{j_1} (\tau) = *$ but $d_{j_{r-l}} \cdots d_{j_1} (\sigma) \neq *$ for some $l < r$.
- right action if there exists a set of maps $d_{j_r} \cdots d_{j_1} = d_{i_r} \cdots d_{i_1}$ with the property that $d_{j_{r-l}} \cdots d_{j_l} (\sigma) = *$ but $d_{j_{r-l}} \cdots d_{j_l} (\tau) \neq *$ for some $l < r$.
- lr action if $\Lambda_{(i_r, n-r)}^{d_{r-1} \cdots d_{k+1} (\omega)}$ is both a right and left action.

Proof. Let σ, τ and ω be the simplices from the proposition above. We will prove the proposition for a left action, but note that an analogous proof works for right actions. We can assume that there exists $1 \leq t < s < r$ and $1 \leq w < r$ so that

$$\begin{aligned} d_{j_s} \cdots d_{j_1}(\sigma) &= * \\ d_{j_{s-1}} \cdots d_{j_1}(\sigma) &\neq * \\ d_{j_t} \cdots d_{j_1}(\tau) &= * \\ d_{j_{t-1}} \cdots d_{j_1}(\tau) &\neq * \end{aligned}$$

and from [2] $\Lambda_{(j_s, n-s)}^{d_{j_{s-1}} \cdots d_{j_1}(\sigma)} = \Lambda_{(j_t, n-t)}^{d_{j_{t-1}} \cdots d_{j_1}(\tau)} = \Lambda_{(i_r, n-r)}^{d_{j_{r-1}} \cdots d_{j_{k+1}}(\omega)}$ which we will simply denote as Λ , but $\delta^{j_1} \cdots \delta^{j_r} f(-) = \delta^{i_1} \cdots \delta^{i_r} f(-)$ and among other elements, we see that on the left we have a_τ acting on $f(-)$ followed by a_σ acting on $a_\tau \cdot f(-)$ and on the right we have $a_\sigma a_\tau$ acting on $f(-)$ so we have $a_\sigma(a_\tau f(-)) = (a_\sigma a_\tau) f(-)$ so Λ is a left action. \square

This gives whether an action Λ is left/right or an lr action from the perspective of each simplex. Recall from [2] that there are many action identifications so to determine if Λ need only be left/right we need to actually consider all action identifications i.e. if $\Lambda^\sigma = \Lambda^\tau$ and Λ^σ is a left action while Λ^τ is a right action, then they are both the same lr action.

Theorem 2.4. *Let X_\bullet be a simplicial set with an NNCMO. Let A be a not necessarily commutative algebra and M be an A -multi-module. Let C be the set of possible distinct actions by X_\bullet and D be the set of actions of A on M . Let $f: C \rightarrow D$ be a map of sets, then $(A, M, X)^\bullet$ exists where each action Λ^σ of A on M is determined by f .*

The proof of this is a straight forward check that the cosimplicial identities hold so it is left to the reader.

In general, given a simplicial set X_\bullet , it seems that it should be a daunting task to determine if X_\bullet admits an NNCMO. In some situations this problem is approachable. We provide two examples that show what types of simplicial sets work and what type of simplicial sets do not work. To start, we consider a special type of simplicial ordering.

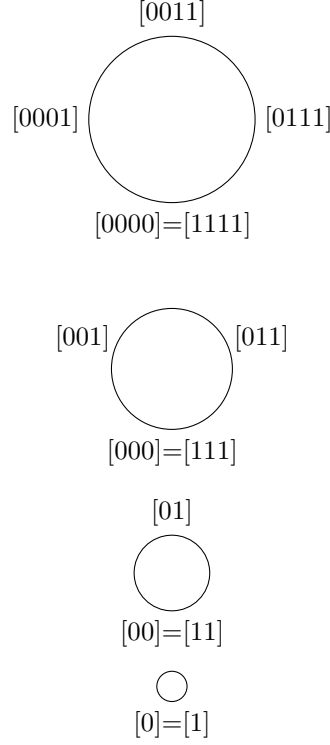
Definition 2.5. We say that a pointed simplicial set X_\bullet has a cyclic ordering if each set $X_n \setminus \{*\}$ has an ordering with the property that if $\sigma, \tau \in X_n \setminus \{*\}$ with $\sigma < \tau$ then $d_i(\sigma) < d_i(\tau)$ for all $0 \leq i \leq n$

Remark 2.6. It is an easy check to see that any set X_\bullet which admits a cyclic ordering also admits an NNCMO, furthermore whether an action is left or right is also simple to see. As can be expected, having a cyclic ordering is a rarity, but it can be seen that the following examples work.

Proposition 2.2. *Let X_\bullet be the minimal simplicial decomposition of $\bigvee_{i \in I} S^1$, then X_\bullet has a cyclic ordering.*

Proof. First notice that X_\bullet has one 1-simplex for each copy of S^1 and one degenerate 1-simplex for the basepoint $*$. The 1-simplices of $X_1 \setminus \{*\}$ can be ordered in any way, so all we need to do is order X_n for larger n . This can be done by giving a cyclic ordering for each sub-simplicial set S_\bullet^1 . One such ordering is

$$S_1^1 : [01]$$

FIGURE 1. Cyclic ordering for S^1_\bullet

$$S^1_2 : [001] < [011]$$

$$S^1_3 : [0001] < [0011] < [0111]$$

$$S^1_4 : [00001] < [00011] < [00111] < [01111]$$

□

We call the ordering above a cyclic ordering because we can actually order all of the S^1 n -simplices (including $*$ clockwise around a circle as is done in Figure 1 (starting with $*$ = $[0\dots 0] = [1\dots 1]$) and see that face maps d_i have the property that $d_i(\sigma) = d_i(\tau)$ if and only if σ and τ are in the $n+1-i$ and $n+2-i$ places around the circle. From this, we can imagine that face maps essentially squeeze adjacent simplices together and do not change the order.

Remark 2.7. In the above ordering, $\Lambda^{[0\dots 01]}$ is a right action, while $\Lambda^{[01\dots 1]}$ is a left action. This demonstrates a fact that we are already aware of—traditional Hochschild cohomology has the ability to work with non-commutative algebras and not necessarily symmetric bi-modules as coefficient modules.

We now turn to an example where X_\bullet does not admit an NNCMO. Consider simplicial sets X_\bullet where X_\bullet contains a 2-simplex which does not contain the base-point and which has no degenerate faces i.e. X_\bullet contains a copy of $[012] \in X_2$, $[01], [02], [12] \in X_1$ and $[0], [1], [2] \in X_0$.

Proposition 2.3. *If X_\bullet is as described above, then X_\bullet does not admit a not necessarily commutative multiplicative ordering.*

Proof. Notice

$$\begin{aligned} d_0([222]) &= d_0([022]) = d_0([122]), \\ d_0([112]) &= d_0([012]), \\ d_1([112]) &= d_1([122]), \\ d_1([002]) &= d_1([022]) = d_1([012]), \\ d_0([22]) &= d_0([12]) = d_0([02]) \\ d_0 d_1 &= d_1 d_0 \end{aligned}$$

so if $[22] < [12]$ then $[022] < [112]$ which implies that $[02] < [12]$. Now, since $[22] < [12]$ we have $[012] > [122]$ but we also have $[02] < [12]$ so $[122] > [012]$ therefore $[12] < [22]$ but this gives a contradiction. \square

3. SECONDARY HOCHSCHILD COHOMOLOGY AND PAIRS OF SIMPLICIAL SETS

In [14] Staic introduced secondary Hochschild cohomology which was used to study B -algebra structures on $A[t]$ given k -algebras A and B with a map $\varepsilon: B \rightarrow A$. In [3] the author and Staic show that secondary Hochschild cohomology is a version of higher order Hochschild cohomology by generalizing Hochschild cohomology to pairs of simplicial sets $X_\bullet \subseteq Y_\bullet$.

To extend noncommutativity to pairs of algebras A and B we first consider the simplicial set Y_\bullet . If Y_\bullet admits an NNCMO, then A and B can both be noncommutative. If Y_\bullet does not admit an NNCMO then B must be commutative. In the second case, we then consider the simplicial set X_\bullet . If X_\bullet admits an NNCMO, then A need not be commutative, however $\varepsilon(B)$ must be in the center of A . For module coefficients, we simply consider where the action comes from in the simplicial set.

ACKNOWLEDGMENT

I would like to thank Andrew Salch and Mihai Staic for conversations concerning this research. I would also like to thank my family for their continued support; in particular, I am grateful to my loving wife Kendall and son Amos.

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